POSITIVE LINEAR EXTENSION OPERATORS FOR SPACES OF AFFINE FUNCTIONS

BY

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ABSTRACT

The following result is proved: let E be an F-space (that is, the space of all continuous affine functions defined on a compact universal cap van:sh'ng at zero) and let $M \subset E$ be an M-ideal. Then, if E/M is a π_1-space with positive defining projections, then there is a positive linear operator ρ: $E/M \to E$ of norm one such that ρ lifts the canonical map $E \to E/M$. In the proof, which heavily depends on work of Ando, we s'udy ensor products of certain convex cones with compact bases, and we calculate the norm of a positive linear operator defined on a finite dimensional space with range in a F-space. Various corollaries are deduced for split faces of compact convex sets and for morphisms of C^*-algebras.

1. Introduction

Let $\tau: A \to B$ be a surjective morphism of C^*-algebras, and consider the problem of finding a linear regular (that is, positive and norm-preserving) right inverse of τ. The well-known fact that c_0 has no closed complement in l^∞ shows that there may be no such operator. On the other hand, the classic result of Borsuk [5] for commutative separable algebras is generalized in Asimow [4] to the effect that there is a solution if B is separable and $\ker \tau$ commutative.

The main result of this paper, Theorem 9, states that a regular linear inverse exists provided B is a π_1-space with positive defining projections. Since the proof is purely order-theoretic, we work in the class of F-spaces, that is, ordered Banach spaces that are representable as spaces of continuous affine functions defined on compact universal caps. In this set-up the right inverse becomes a regular linear extension operator for the natural restriction map.

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Basic for our work is Ando's paper [3] in which he introduced the concept of a splittable convex set relative to an M-ideal, thereby generalizing the M-ideals of Alfsen and Effros [1] and proving extension theorems analogous to [1, Th. 5.4, Part I]. These results and the method of stepwise approximate extension devised by Michael and Pelczynski [8] enabled him to show the existence of a norm-preserving right inverse of $\tau: E \to E/M$ where $M \subset E$ is an M-ideal in the Banach space E such that E/M is a π_1-space. In addition, his paper contains a version specialized to ordered Banach spaces with simplicial positive cones in the approximating finite dimensional subspaces of E/M.

Our result is a modification of the latter result in the sense that we may dispense with the condition of simplicial cones (which we must do in the case of non-commutative C^*-algebras due to their lack of peaked partitions, see [3] and [8]) when specializing to spaces of affine functions.

In the process of proving the theorem we study tensor products of certain w^*-closed convex sets using techniques going back to Grothendieck [7]. We obtain some results which may have independent interest (Theorem 11).

With pleasure I thank T. B. Andersen for many stimulating discussions on linear extension operators and for relevant references.

2. Preliminaries

Unless otherwise stated we consider only real Banach spaces. We use standard notation. E^* is the dual of E, $L(F, E)$ denotes the set of bounded linear operators $F \to E$, and we use freely the injections of $E \otimes F$ into $L(E^*, F)$ and $L(F^*, E)$, and into $L(H, E)$ if F happens to be the dual of H. The projective (respectively, least dual) cross-norm is denoted by γ (respectively, λ). We use $(\cdot)^-$ (respectively, $(\cdot)^\sim$) for norm- (respectively, w^*-) closure. In particular, if F is finite dimensional, $(\cdot)^\sim$ denotes closure for the topology of pointwise $\sigma(E^{**}, E^*)$ convergence on $L(F, E^{**})$ since this space is the dual of $F \otimes_\gamma E^*$.

If S is a subset of the Banach space E, S_1 denotes the intersection of S and the closed unit ball of E. By abuse of notation we write $S \otimes K$ for $\{e \otimes f \mid e \in S, f \in K\}$ if $K \subset F$.

If K is a subset of the finite dimensional space F and S a subset of E we define, following [3],

$$G(K, S) = \{\phi \in L(F, E) \mid \phi(K) \subset S\},$$

$$\underline{G}(K, \tilde{S}) = \{\phi \in L(F, E^{**}) \mid \phi(K) \subset \tilde{S}\}.$$

Clearly one has $G(K,S)^{\sim} \subset \underline{G}(K,\tilde{S})$; it is an important problem to find conditions for equality.

We recall that a closed subspace M of the Banach space E is called an M-ideal, if $M^0 \subset E^*$ has a closed complement N such that $E^* \simeq M^0 \oplus N$ is an l^1 direct sum. It is shown in [1], that N is unique, if it exists.

We can now state the result of Ando [3]. For information on π_1-spaces, see [8].

THEOREM 1. *Let E be an ordered Banach space and $M \subset E$ an M-ideal such that the defining projection P onto M^0 satisfies $0 \leqq P \leqq I$. Assume furthermore that E/M is a π_1-space such that the defining projections onto the finite dimensional subspaces $\{F_n\}$ are positive. Then, if*

$$G(F_n^+,E^+)^{\sim} = \underline{G}(F_n^+,(E^{**})^+) \quad \textit{for all } n, \tag{1}$$

there exists a linear regular map $\rho: E/M \to E$ such that $\tau \circ \rho$ is the identity on E/M.

This is [3, Lemma 9]. If we write out the result of the first step in the proof of Theorem 1 we obtain Theorem 2.

THEOREM 2. *Let M and E be as above. Let F be a finite dimensional ordered Banach space, and $\psi: F \to E/M$ a positive linear map. Then, if*

$$G(F^+,E^+)^{\sim} = \underline{G}(F^+,(E^{**})^+),$$

there exists a positive linear map $\phi: F \to E$ such that $\| \phi \| = \| \psi \|$ and $\tau \circ \rho = \psi$.

3. Tensor products of certain convex sets

In this section E is a Banach space and F a finite dimensional Banach space. We shall be computing polars in various dualities. We write $(\ )^{\circ}$ for a polar computed in the dual space, and $(\ )^{\bullet}$ for the polar computed in the predual.

PROPOSITION 3. *Let $K \subset F$ and $S \subset E$ be norm-closed, convex sets containing zero. Then the following are equivalent:*

(i) $G(K,S)^{\sim} = \underline{G}(K,\tilde{S})$ *and*

(ii) $\widetilde{\mathrm{co}}(K \otimes S^{\circ}) = \overline{\mathrm{co}}(K \otimes S^{\circ})$.

PROOF. Let us compute $G(K,S)^{\sim} = G(K S)^{\circ\circ}$. With $\phi = \sum f_i^* \otimes e_i \in F^* \otimes E$ we have $\phi \in G(K,S)$ if and only if

$$\phi(f) = \sum \langle f_i^*, f \rangle e_i \in S \qquad \text{for all } f \in K.$$

Since $S = S^{\circ\bullet}$, this is equivalent to

$$\langle \phi(f), e^* \rangle \leqq 1 \quad \text{for all } f \in K,\ e^* \in S^\circ,$$

and therefore, $G(K,S) = (K \otimes S^\circ)^\bullet$, and we obtain

$$G(K,S)^{\circ\circ} = (K \otimes S^\circ)^{\bullet\circ\circ} = (\tilde{\text{co}}(K \otimes S^\circ))^\circ. \tag{2}$$

By analogous reasoning we obtain $\underline{G}(K,\tilde{S}) = (K \otimes S^\circ)^\circ$, and consequently

$$\underline{G}(K,\tilde{S}) = (\overline{\text{co}}(K \otimes S^\circ))^\circ. \tag{3}$$

Since $\underline{G}(K,\tilde{S})$ and $G(K,S)^\sim$ coincide if and only if their polars in $F \otimes E^*$ do, the equivalence of (i) and (ii) follows from equations (2) and (3).

PROPOSITION 4. *The conditions of Proposition 3 hold if K and S° are w^*-compact.*

PROOF. Let $\phi \in \tilde{\text{co}}(K \otimes S^\circ)$. We must show that $\phi \in \overline{\text{co}}(K \otimes S^\circ)$. By [7, Prop. 27] ϕ is a weak integral of a Radon probability measure μ on $K \times S^\circ$ for the duality $\langle F^* \otimes E, F \otimes E^* \rangle$. Indeed, μ can be obtained as a w^*-limit point of discrete measures $\sum \lambda_i \delta_{f_i} \otimes \delta_{e_i^*}$ with $\sum \lambda_i f_i \otimes e_i^*$ w^*-approximating ϕ. We have

$$\langle \phi, f^* \otimes e \rangle = \int_{K \times S^\circ} \langle f \otimes e^*, f^* \otimes e \rangle \, d\mu(f, e^*).$$

By compactness, we can cover K by a finite set $\{U_i\}_{i=1}^n$ of closed convex neighbourhoods of diameter less than ε. Pick points $f_i \in U_i$, and turn $\{U_i\}$ into a partition by defining $V_1 = U_1$, $V_i = U_i \setminus V_{i-1}$ for $2 \leqq i \leqq n$. We disregard indices with $\mu(V_i \times S^\circ) = 0$. Define $\psi_i \in F \otimes E^*$ by

$$\mu(V_i \times S^\circ)\psi_i = \int_{V_i \times S^\circ} f_i \otimes e^* d\mu(f, e^*).$$

Since

$$\begin{aligned} \langle \mu(V_i \times S^\circ)\psi_i, f^* \otimes e \rangle &= \int_{V_i \times S^\circ} \langle f_i, f^* \rangle \langle e^*, e \rangle d\mu(f, e^*) \\ &= \langle f_i, f^* \rangle \int_{V_i \times S^\circ} \langle e^*, e \rangle d\mu(f, e^*) \end{aligned}$$

we see that $\psi_i = f_i \otimes e_i^*$ where e_i^* is the barycenter of the measure that arises as the transform of $(\mu(V_i \times S^\circ))^{-1} \mu_{|V_i \times S^\circ}$ under the projection $K \times S^\circ \to S^\circ$. In particular $e_i^* \in S^\circ$, and therefore

$$\psi = \sum \mu(V_i \times S^\circ) f_i \otimes e_i^* \in \text{co}(K \otimes S^\circ).$$

Since $f_i \in U_i$ and $\operatorname{diam}(V_i) < \varepsilon$, we have $\| f - f_i \| < \varepsilon$ for $f \in V_i \subset U_i$. We now derive the following estimate:

$$\begin{aligned} |\langle \phi - \psi, f^* \otimes e \rangle| &= \left| \sum \int_{V_i \times S^\circ} \langle f - f_i, f^* \rangle \langle e^*, e \rangle \, d\mu \, (f, e^*) \right| \\ &\leqq (\sup \| e^* \|) \cdot \| e \| \sum \int_{V_i \times S^\circ} |\langle f - f_i, f^* \rangle| \, d\mu(f, e^*) \\ &\leqq (\sup \| e^* \|) \| e \| \sum \varepsilon \| f^* \| \mu(V_i \times S^\circ) \\ &= (\sup \| e^* \|) \cdot \varepsilon \cdot \| e \| \cdot \| f^* \|, \end{aligned}$$

where the supremum is taken over S°. Hence $\| \phi - \psi \|_\lambda \leqq \varepsilon \cdot \sup \| e^* \|$, whence the desired norm approximation of ϕ by ψ.

COROLLARY 5. *$(F^* \otimes_\lambda E)^*$ is isometric to $F \otimes_\gamma E^*$.*

PROOF. We apply Proposition 4 with $K = F_1$ (unit ball of F) and $S = E_1$ to obtain

$$\widetilde{\mathrm{co}}(F_1 \otimes E_1^\circ) = \overline{\mathrm{co}}(F_1 \otimes E_1^\circ).$$

E_1° is the unit ball in E^*; the set on the left-hand side is the polar of the unit ball in $F^* \otimes_\lambda E$, while the set of the right-hand side is the unit ball in $F \otimes_\gamma E^*$. This shows that γ is the dual norm of λ. Q.E.D.

REMARK. Notice that in the proof of Corollary 5 we only used the fact that $(F^* \otimes_\lambda E)^*$ and $F \otimes_\gamma E^*$ are isomorphic. The corollary holds with much weaker assumptions on F than finite dimensionality (see [7, Th. 8]), but the proof presented here in the finite dimensional case is considerably shorter than that in [7].

The next proposition deals with convex sets that are not compact.

PROPOSITION 6. *Let K and S be as in Proposition 3 and suppose in addition that K and S are closed convex cones such that*

$$C \cdot \mathrm{co}(K_1 \otimes (S^\circ)_1) \supset (\mathrm{co}(K \otimes S^\circ))_1$$

for some $C > 0$. Then the conditions of Proposition 3 hold.

PROOF. By the Krein-Šmulian theorem it suffices to prove that $\overline{\mathrm{co}}(K \otimes S^\circ)$ meets the unit ball U of $F \otimes E^*$ in a w^*-closed set. By assumption we have

$$U \cap \mathrm{co}(K \otimes S^\circ) \subset C \cdot \mathrm{co}(K_1 \otimes (S^\circ)_1)$$

whence, since $\widetilde{\mathrm{co}}(K_1 \otimes (S^\circ)_1) = \overline{\mathrm{co}}(K_1 \otimes (S^\circ)_1)$ by Proposition 4,

$$(U \cap \overline{\mathrm{co}}(K \otimes S^\circ))^\sim = (U \cap \mathrm{co}(K \otimes S^\circ))^\sim \subset (C \cdot \mathrm{co}(K_1 \otimes (S^\circ)_1))^\sim \cap U$$
$$= (C \cdot \overline{\mathrm{co}}(K_1 \otimes (S^\circ)_1)) \cap U \subset \overline{\mathrm{co}}(K \otimes S^\circ) \cap U$$

from which it follows that $U \cap \overline{\mathrm{co}}(K \otimes S^\circ)$ is w^*-closed. Q.E.D.

4. Extension operators for spaces of affine functions

The key result of this section is Proposition 8 which states that (1) holds when E is an F-space. For general information on F-spaces, see, for example, [1]. We recall that an F-space is an ordered Banach space of the form $A_0(X)$, where X is a compact universal cap of a cone. We shall need the following two properties of F-spaces:

(i) If $\{e_i\}$ is a finite subset of the open ball unit, then there is an e in the open unit ball with $e \geqq e_i$ (the unit ball is approximately directed).

(ii) If $e^* \in E^*$ with $e^* \geqq 0$ then $\| e^* \| = \sup \langle e, e^* \rangle$ where the supremum is taken over the positive part of the open unit ball.

Finally, we shall need the following (well-known) lemma.

LEMMA 7. *Let F be a finite dimensional, ordered Banach space. Then there is a $C > 0$ such that if $f_1, \cdots, f_n \in F^+$, then*

$$C \| \Sigma f_i \| \geqq \Sigma \| f_i \|.$$

PROOF. We may assume $\Sigma \| f_i \| = 1$ by homogeneity. The set $A = \{ f \in F^+ \mid \| f \| = 1\}$ is closed and its convex hull does not contain zero (it is tacitly assumed that F^+ is closed and proper). Since F is finite dimensional, $\mathrm{co}(A)$ is closed, and therefore there is a ball around zero with radius, say C^{-1}, not meeting $\mathrm{co}(A)$. But $\Sigma f_i \in \mathrm{co}(A)$, so we are done.

PROPOSITION 8. *Let E be a F-space and F a finite dimensional, ordered Banach space. Then*

$$G(F^+, E^+)^\sim = \underline{G}(F^+, (E^+)^\sim).$$

PROOF. By Proposition 6 it suffices to verify that $C \cdot \mathrm{co}(F_1^+ \otimes (E^+)^\circ{}_1) \supset (\mathrm{co}(F^+ \otimes (E^+)^\circ))_1$ for some $C > 0$. Since $(E^+)^\circ = -E^{*+}$, we may replace $(E^+)^\circ$ by E^{*+}. Let $\phi = \Sigma f_i \otimes e_i^* \in (\mathrm{co}(F^+ \otimes E^{*+}))_1$. Considering ϕ as an element of $L(E,F)$, ϕ is positive and

$$\| \phi \| = \sup_{\|e\| < 1} \| \phi(e) \| \geqq \sup \{ \| \phi(e) \| \mid e \geqq 0, \| e \| < 1 \}.$$

By Lemma 7 we see that

$$\| \phi(e) \| = \| \Sigma \langle e, e_1^* \rangle f_i \| \geqq C^{-1} \Sigma \langle e, e_i^* \rangle \| f_i \|, \ e \geqq 0$$

for some $C > 0$. Hence

$$\| \phi \| \geqq C^{-1} \sup_e \Sigma \ \langle e, e_i^* \rangle \| f_i \|,$$

where the supremum is taken over $\| e \| < 1$, $e \geqq 0$. The latter quantity equals $C^{-1} \Sigma \| e_i^* \| \| f_i \|$ by (i) and (ii), and we conclude that $\Sigma \| e_i^* \| \| f_i \| \leqq C$. This obviously implies that $\phi \in C \cdot \mathrm{co}(F_1^+ \otimes E_1^{*+})$. Thus the proposition is proved.

Now we can prove the main result.

THEOREM 9. *Let E be an F-space, $M \subset E$ an M-ideal and $\tau: E \to E/M$ the natural map. Suppose that E/M is a π_1-space such that the defining projections are positive. Then there is a regular linear map $\rho: E/M \to E$ such that $\tau \circ \rho$ is the identity on E/M.*

PROOF. We shall apply Theorem 1. Condition (1) holds, by Proposition 8, and the projection P satisfies $0 \leqq P \leqq I$ by [1, p. 164]. (In fact, P^* is a restriction operator when E^{**} and $(E/M)^{**}$ are identified with spaces of affine functions.)

By duality we obtain Theorem 10.

THEOREM 10. *Let X be a compact convex set in a locally convex real vector space. Let $Y \subset X$ be a closed split face such that $A(Y)$ is a π_1-space with positive defining projections. Then there exists a continuous affine retraction $r: X \to Y$.*

PROOF. The kernel of the restriction map $A(X) \to A(Y)$ is an M-ideal (see [1]), so we apply Theorem 9 to find a regular extension operator $\rho: A(Y) \to A(X)$. With δ any state on $A(Y)$ we let $\rho' = \rho + (1 - \rho(1))\,\delta: A(Y) \to A(X)$. It is verified by inspection that ρ' is a regular extension operator with $\rho'(1) = 1$. The transpose of ρ' is the desired retraction.

THEOREM 11. *Let E be an F-space and F an ordered finite dimensional Banach space. Then*

(i) *the set of positive linear maps $F \to E$ of norm at most 1 is w^*-dense in the set of positive linear maps $F \to E^{**}$ of norm at most one, and*

(ii) *if $M \subset E$ is an M-ideal and $\phi: F \to E/M$ is positive with $\| \phi \| = 1$, then there is a positive linear map $\psi: F \to E$ such that $\| \psi \| = 1$ and $\tau \circ \psi = \phi$.*

PROOF. (i) Proposition 8 yields $G(F^+, E^+)^{\sim} = \underline{G}(F^+, E^{**+})$ since $(E^+)^{\sim} = E^{**+}$. As observed by Ando [3] we have $(G(F^+, E^+)_1)^{\sim} = ((G(F^+, E^+)^{\sim})_1$

because $G(F^+, E^+)$ is a closed cone. Hence (i) is proved. (ii) Again Proposition 8 applies, this time to Theorem 2 to give the conclusion.

REMARK. A result analogous to (ii) has been proved by T. B. Andersen (unpublished).

The fact that the approximation property or some stronger property is always assumed (see [2], [3], [6], and [8]) for some Banach space in extension theorems is explained by Davie [6]. His construction can easily be modified to yield the following proposition.

PROPOSITION 12. *Let $\hat{E}$ be a separable F-space. Then there exists an F-space E and a quotient map $\tau: E \to \hat{E}$ such that* $\ker \tau$ *is an M-ideal and E has the metric approximation property with positive defining operators. Thus the existence of a regular extension operator would imply that $\hat{E}$ has the metric approximation property with positive defining operators.*

SKETCH OF PROOF. Consider $\hat{E}$ as the space of continuous affine functions on its state space, vanishing at zero. Following the procedure of [6] we can imbed this state space in a compact subset S of R^∞. Now, letting $X = \overline{\text{co}}(S)$, $E = A_0(X)$ will do.

REMARK. We have been unable to prove Theorem 9 with the weaker assumption of merely positive metric approximation on E/M. The question whether the *positive* π_1-property is necessary is left open too.

5. Right inverses for morphism of C^*-algebras

In this section we return to the problem originally asked. To extend Theorem 9 to C^*-algebras we need the following lemma.

LEMMA 13. *Let $\rho: B \to A$ be a positive linear map of C^*-algebras. Then*

$$\| \rho_{|B_h} \| = \| \rho \|,$$

where B_h is the hermitian part of B.

PROOF. If B is unital this is essentially [9, Cor. 1]. If not, we can extend ρ to the algebras with units adjoined by letting $\tilde{\rho}(1) = \lambda \cdot 1$, where $\lambda = \| \rho_{|B_h} \|$. It is a routine matter to check that $\tilde{\rho}$ is positive, and the result follows.

THEOREM 14. *Let $\tau: A \to B$ be a surjective morphism of C^*-algebras. Suppose B is a π_1-space such that the defining projections are positive. Then there* *is a positive linear map ρ: $B \to A$ of norm one such that $\tau \circ \rho$ is the identity on B.*

PROOF. The hermitian part A_h (respectively, B_h) of A (respectively, B) is an F-space, $M = \ker\tau \cap A_h$ is an M-ideal [1] so we may apply Theorem 9 to find a regular linear map ρ': $B_h \to A_h$ such that $\tau \circ \rho' = \mathrm{id}_{B_h}$. The complex extension ρ of ρ' to B meets all the conditions in view of Lemma 13.

Theorem 10 and Theorem 11(i) obviously apply to C^*-algebras. However, it is not clear whether the map ψ of Theorem 11(ii) has norm 1 when extended to a complex linear map.

An interesting problem is to find the class of C^*-algebras that are π_1-spaces with positive projections. Commutative algebras, UHF algebras, and the algebra of compact operators on separable Hilbert space are in that class, and probably a large class of CCR algebras, too.

We know of no algebras outside the class, let alone algebras without the approximation property. It has been conjectured that the C^*-algebra of the free group with two generators does not have the approximation property.

ADDED IN PROOF

T. B. Anderson has improved Theorem 9 in the unital case [10].

REFERENCES

1. E. Alfsen and E. Effros, *Structure in real Banach spaces*, Ann. of Math. **96** (1972), 98–173.

2. T. B. Andersen, *On Banach space valued extensions from split faces*, Pacific J. Math. **42** (1971), 1–9.

3. T. Ando, *Closed range theorems for convex sets and linear lifitings*, Pacific J. Math. **44** (1973), 393–410.

4. L. Asimow, *Affine selections on simplicially split compact convex sets*, J. London Math. Soc. (2) **6** (1973), 533–538.

5. K. Borsuk, *Über Isomorphie der Funktionalräume*, Bull. Int. Acad. Pol. Sci. 1933, 1–10.

6. A. M. Davie, *Linear extension operators for spaces and algebras of functions*, Amer. J. Math. **94** (1972), 156–172.

7. A. Grothendieck, *Produits tensorielles topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

8. E. Michael and A. Pelczynski, *A linear extension theorem*, Illinois J. Math **11** (1967), 563–579.

9. B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J. **33** (1966), 413–416.

10. T. B. Anderson, *Linear extensions, projections and split faces*, preprint, Arhus 1973 (to appear in J. Functional Analysis).

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